

CLUSTERING PHENOMENA FOR LINEAR PERTURBATION OF THE YAMABE EQUATION

ANGELA PISTOIA AND GIUSI VAIRA

This paper is warmly dedicated to Professor Abbas Bahri on the occasion of his 60th birthday

ABSTRACT. Let (M, g) be a non-locally conformally flat compact Riemannian manifold with dimension $N \geq 7$. We are interested in finding positive solutions to the linear perturbation of the Yamabe problem

$$-\mathcal{L}_g u + \epsilon u = u^{\frac{N+2}{N-2}} \text{ in } (M, g)$$

where the first eigenvalue of the conformal laplacian $-\mathcal{L}_g$ is positive and ϵ is a small positive parameter. We prove that for any point $\xi_0 \in M$ which is non-degenerate and non-vanishing minimum point of the Weyl's tensor and for any integer k there exists a family of solutions developing k peaks collapsing at ξ_0 as ϵ goes to zero. In particular, ξ_0 is a non-isolated blow-up point.

Keywords: Yamabe problem, linear perturbation, blow-up points

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1. INTRODUCTION

Let (M, g) be a smooth, compact Riemannian manifold of dimension $N \geq 3$. The Yamabe problem consists in finding metrics of constant scalar curvature in the conformal class of g . It is equivalent to finding a positive solution to the problem

$$\mathcal{L}_g u + \kappa u^{\frac{N+2}{N-2}} = 0 \text{ in } M, \quad (1.1)$$

for some constant κ . Here $\mathcal{L}_g u := \Delta_g u + \frac{N-2}{4(N-1)} R_g u$ is the conformal laplacian, Δ_g is the Laplace-Beltrami operator and R_g is the scalar curvature of the manifold.

In particular, if u solves (1.1), then the scalar curvature of the metric $\tilde{g} = u^{\frac{4}{N-2}} g$ is nothing but $\frac{4(N-1)}{N-2} \kappa$. Yamabe problem has been completely solved by Yamabe [26], Aubin [1], Trudinger [25] and Schoen [20] (see also the proof given by Bahri [2]). The solution is unique in the case of negative scalar curvature and it is unique (up to a constant factor) in the case of zero scalar curvature. The uniqueness is not true anymore in the case of positive scalar curvature. Indeed, Schoen [21] and Pollack in [16] exhibit examples where a large number of high energy solutions with high Morse index exist. Thus it is natural to ask if the set of solutions is compact or not as it was raised by Schoen in [22]. It is also useful to point out that in the case of the round sphere (\mathbb{S}^N, g_0) the compactness does not hold (see Obata in [15]). Indeed, the scalar curvature $R_{g_0} = N(N-1)$ and the Yamabe problem (1.1) reads as

$$-\Delta_{g_0} u + \frac{N(N-2)}{4} u = u^{\frac{N+2}{N-2}} \text{ in } (\mathbb{S}^N, g_0)$$

which is equivalent (via the stereographic projection) to the equation in the Euclidean space

$$-\Delta U = U^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N. \quad (1.2)$$

It is known that (1.2) has infinitely many solutions, the so called *standard bubbles*,

$$U_{\mu, y}(x) = \mu^{-\frac{N-2}{2}} U\left(\frac{x-y}{\mu}\right), \quad x, y \in \mathbb{R}^N, \quad \mu > 0, \quad \text{where } U(x) := \alpha_N \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}. \quad (1.3)$$

Here $\alpha_N := N(N-2)^{\frac{N-2}{4}}$.

The compactness turns out to be true when the dimension of the manifold satisfies $3 \leq N \leq 24$ as it was shown by Khuri, Marques and Schoen [9]) (previous results were obtained by Schoen [23], Schoen and Zhang [24], Li and Zhu [12], Li and Zhang [11], Marques [13] and Druet [6]), while it is false when $N \geq 25$ thanks to the examples built by Brendle [4] and Brendle and Marques [5]. The proof of compactness strongly relies on proving sharp pointwise estimates at a blow-up point of the solution. In particular, when compactness holds every sequence of unbounded solutions to (1.1) must blow-up at some points of

the manifold which are necessarily isolated and simple, i.e. around each blow-up point ξ_0 the solution can be approximated by a standard bubble (see (1.3))

$$u_n(x) \sim \alpha_N \frac{\mu_n^{\frac{N-2}{2}}}{(\mu_n^2 + (d_g(x, \xi_n))^2)^{\frac{N-2}{2}}} \quad \text{for some } \xi_n \rightarrow \xi_0 \text{ and } \mu_n \rightarrow 0.$$

More precisely, let u_n be a sequence of solutions to problem (1.1). We say that u_n blows-up at a point $\xi_0 \in M$ if there exists $\xi_n \in M$ such that $\xi_n \rightarrow \xi_0$ and $u_n(\xi_n) \rightarrow +\infty$. ξ_0 is said to be a *blow-up point* for u_n . Blow-up points can be classified according to the definitions introduced by Schoen in [22]. $\xi_0 \in M$ is an *isolated blow-up point* for u_n if there exists $\xi_n \in M$ such that ξ_n is a local maximum of u_n , $\xi_n \rightarrow \xi_0$, $u_n(\xi_n) \rightarrow +\infty$ and there exist $c > 0$ and $R > 0$ such that

$$0 < u_n(x) \leq c \frac{1}{d_g(x, \xi_n)^{\frac{N-2}{2}}} \quad \text{for any } x \in B(\xi_0, R).$$

Moreover, $\xi_0 \in M$ is an *isolated simple blow-up point* for u_n if the function

$$\hat{u}_n(r) := r^{\frac{N-2}{2}} \frac{1}{|\partial B(\xi_n, r)|_g} \int_{\partial B(\xi_n, r)} u_n d\sigma_g$$

has a exactly one critical point in $(0, R)$.

Motivated by the previous consideration, we are led to study the linear perturbation of the Yamabe problem

$$-\mathcal{L}_g u + \epsilon u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } (M, g) \quad (1.4)$$

where the first eigenvalue of $-\mathcal{L}_g$ is positive and ϵ is a small parameter. In particular, we address the following questions.

- (i) Do there exist solutions to (1.4) which blow-up as $\epsilon \rightarrow 0$?
- (ii) Do there exist solutions to (1.4) with non-isolated blow-up points, namely with *clustering* blow-up points?
- (iii) Do there exist solutions to (1.4) with non-isolated simple blow-up points, namely with *towering* blow-up points?

Concerning question (i), Druet in [6] proved that equation (1.4) does not have any blowing-up solution when $\epsilon < 0$ and $N = 3, 4, 5$ (except when the manifold is conformally equivalent to the round sphere). It is completely open the case when the dimension is $N \geq 6$. The situation is completely different when $\epsilon > 0$. Indeed, if $N = 3$ no blowing-up solutions exist as proved by Li-Zhu [12], while if $m \geq 4$ blowing-up solutions do exist as shown by Esposito, Pistoia and Vetois in [8]. In particular, if the dimension $N \geq 6$ and the manifold is not locally conformally flat, Esposito, Pistoia and Vetois built solutions which blow-up at non-vanishing stable critical points ξ_0 of the Weyl's tensor, i.e. $|\text{Weyl}_g(\xi_0)|_g \neq 0$. In this paper, we show that the blowing-up point ξ_0 is not-isolated as soon as it is a non-degenerate minimum point of the Weyl's tensor. This result gives a positive answer to question (ii). Finally, a positive answer to question (iii) has been giving by Morabito, Pistoia and Vaira in a forthcoming paper [14].

Now, let us state the main result obtained in this paper.

Theorem 1.1. *Let (M, g) be not locally conformally flat and $N \geq 7$. Let $\xi_0 \in M$ be a non-degenerate minimum point of $\xi \rightarrow |\text{Weyl}_g(\xi)|_g^2$. Then, for any $k \in \mathbb{N}$, there exist $\xi_\epsilon^j \in M$ for $j = 1, \dots, k$ and $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$ the problem (1.4) has a solution $(u_\varepsilon)_\varepsilon$ with k positive peaks at ξ_ϵ^j and $\xi_\epsilon^j \rightarrow \xi_0$ as $\varepsilon \rightarrow 0$.*

Let us point out that Robert and Vetois in [18] built solutions having clustering blow-up points for a special class of perturbed Yamabe type equations which look like

$$-\mathcal{L}_g u + \epsilon H u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } (M, g). \quad (1.5)$$

where the potential H is chosen with k distinct strict local maxima concentrating at a point ξ_0 with $|\text{Weyl}_g(\xi_0)|_g \neq 0$. Indeed, these maxima points generate solutions with k positive peaks collapsing to ξ_0 as ϵ goes to zero. Their result is related to a suitable choice of the potential H , but actually our result shows that the clustering phenomena is intrinsic in the geometry of the manifold.

Let us give an example. The warped product $(S^n \times S^m, g_{S^n} \otimes f^2 g_{S^m})$, is the Riemannian manifold $S^n \times S^m$ equipped with the metric $g = g_{S^n} \otimes f^2 g_{S^m}$. Here $f : S^n \rightarrow \mathbb{R}$ is a positive function called warping function. It is easy to see that if the warping function $f \equiv 1$ than the the product manifold $(S^n \times S^m, g_{S^n} \otimes g_{S^m})$ has Weyl tensor different from zero at any point. Using similar arguments to the ones used in [17], we can prove that for generic warping functions f close to the constant 1, the Weyl tensor has a non-degenerate and non-vanishing minimum point.

The proof of our result relies on a finite dimensional Ljapunov-Schmidt reduction, whose main steps are described in Section 3 and their proofs are postponed in Section 4. Section 2 is devoted to recall some known results.

2. PRELIMINARIES

We provide the Sobolev space $H_g^1(M)$ with the scalar product

$$\langle u, v \rangle = \int_M \langle \nabla u, \nabla v \rangle_g d\nu_g + \beta_N \int_M R_g u v d\nu_g \quad (2.1)$$

where $d\nu_g$ is the volume element of the manifold. Here $\beta_N := \frac{N-2}{4(N-1)}$. We let $\|\cdot\|$ be the norm induced by $\langle \cdot, \cdot \rangle$. Moreover, for any function u in $L^q(M)$, we denote the L^q -norm of u by $\|u\|_q = (\int_M |u|^q d\nu_g)^{1/q}$.

We let $\mathfrak{r}^* : L^{\frac{2N}{N+2}}(M) \rightarrow H_g^1(M)$ be the adjoint operator of the embedding $\mathfrak{r} : H_g^1(M) \hookrightarrow L^{2^*}(M)$, i.e. for any w in $L^{\frac{2N}{N+2}}(M)$, the function $u = \mathfrak{r}^*(w)$ in $H_g^1(M)$ is the unique solution of the equation $-\Delta_g u + \beta_N R_g u = w$ in M . By the continuity of the embedding of $H_g^1(M)$ into $L^{2^*}(M)$, we get

$$\|\mathfrak{r}^*(w)\| \leq C \|w\|_{\frac{2N}{N+2}} \quad (2.2)$$

for some positive constant C independent of w . We rewrite problem (1.4) as

$$u = \mathfrak{r}^*[f(u) - \varepsilon u], \quad u \in H_g^1(M) \quad (2.3)$$

where we set $f(u) := (u^+)^p$ with $p = \frac{N+2}{N-2}$.

We also define the energy $J_\varepsilon : H_g^1(M) \rightarrow \mathbb{R}$

$$J_\varepsilon(u) := \frac{1}{2} \int_M (|\nabla_g u|^2 + \beta_N R_g u^2 + \varepsilon u^2) d\nu_g - \frac{1}{p+1} \int_M (u^+)^{p+1} d\nu_g, \quad (2.4)$$

whose critical points are solutions to the problem (1.4).

We are going to read the euclidean bubble defined in (1.3) on the manifold via a geodesic normal coordinate system around a point $\xi \in M$, i.e.

$$\mathcal{U}_{\mu, \xi}(z) = U_{\mu, 0} \left(\exp_\xi^{-1}(z) \right) = \mu^{-\frac{N-2}{2}} U \left(\frac{\exp_\xi^{-1}(z)}{\mu} \right), \quad z \in B_g(\xi, r).$$

It is necessary to write the conformal laplacian in geodesic normal coordinates around the point ξ . In particular, if $x \in B(0, r)$ using standard properties of the exponential map we can write

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij}) \partial_{ij}^2 u + g^{ij} \Gamma_{ij}^k \partial_k u, \quad (2.5)$$

with

$$g^{ij}(x) = \delta^{ij}(x) - \frac{1}{3} R_{iabj}(\xi) x_a x_b + O(|x|^3) \text{ and } g^{ij}(x) \Gamma_{ij}^k(x) = \partial_l \Gamma_{ii}^k(\xi) x_l + O(|x|^2). \quad (2.6)$$

Here R_{iabj} denotes the Riemann curvature tensor and Γ_{ii}^k the Christoffel's symbols. Therefore, if we compare the conformal laplacian with the euclidean laplacian of the bubble the error at main order looks like

$$\mathcal{L}_g \mathcal{U}_{\mu, \xi} - \Delta \mathcal{U}_{\mu, \xi} \sim -\frac{1}{3} \sum_{a,b,i,j=1}^N R_{iabj}(\xi) x_a x_b \partial_{ij}^2 U_{\mu, 0} + \sum_{i,l,k=1}^N \partial_l \Gamma_{ii}^k(\xi) x_l \partial_k U_{\mu, 0} + \beta_N R_g(\xi) U_{\mu, 0}.$$

For later purposes, it is necessary to kill this main term by adding to the bubble an higher order term V which is defined as follows. First, we remind that any solution of the linear equation (see [3])

$$-\Delta v = p U^{p-1} v \quad \text{in } \mathbb{R}^N, \quad (2.7)$$

is a linear combination of the functions

$$\psi^0(x) = x \cdot \nabla U(x) + \frac{N-2}{2}U(x) \text{ and } \psi^i(x) = \partial_i U(x), \quad i = 1, \dots, N. \quad (2.8)$$

Next, we introduce the higher order term V which has been built in Section 2 in [7].

Proposition 2.1. *For any point $\xi \in M$, there exist $\nu(\xi) \in \mathbb{R}$ and a function $V \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ solution to*

$$-\Delta V - f'(U)V = - \sum_{a,b,i,j=1}^N \frac{1}{3} R_{iabj}(\xi) x_a x_b \partial_{ij}^2 U + \sum_{i,l,k=1}^N \partial_l \Gamma_{ii}^k(\xi) x_l \partial_k U + \beta_N R_g(\xi) U + \nu(\xi) \psi^0 \text{ in } \mathbb{R}^N, \quad (2.9)$$

with

$$\int_{\mathbb{R}^N} V(x) \psi^i(x) dx = 0, \quad i = 0, 1, \dots, N.$$

Moreover, there exists $C \in \mathbb{R}$ such that

$$|V(x)| + |x| |\partial_k V(x)| + |x|^2 |\partial_{ij}^2 V(x)| \leq C \frac{1}{(1 + |x|^2)^{\frac{N-4}{2}}}, \quad x \in \mathbb{R}^N. \quad (2.10)$$

3. CLUSTERING

3.1. The ansatz: the cluster. Let r_0 be a positive real number less than the injectivity radius of M and χ be a smooth cut-off function such that $0 \leq \chi \leq 1$ in \mathbb{R} , $\chi \equiv 1$ in $[-r_0/2, r_0/2]$, and $\chi \equiv 0$ out of $[-r_0, r_0]$. Let also η be a smooth cutoff function such that $0 \leq \eta \leq 1$ in \mathbb{R} , $\eta \equiv 1$ in $[-1, 1]$, and $\eta \equiv 0$ out of $[-2, 2]$.

Let $k \geq 1$ be a fixed integer. Assume that $\xi_0 \in M$ is a non degenerate minimum point of $\xi \rightarrow |\text{Weyl}_g(\xi)|_g^2$ with $|\text{Weyl}_g(\xi_0)| \neq 0$, i.e.

$$\nabla_g |\text{Weyl}_g(\xi_0)|_g^2 = 0 \text{ and the quadratic form } \mathcal{Q}(\xi_0) := D_g^2 |\text{Weyl}_g(\xi_0)|_g^2 \text{ is positive definite.} \quad (3.1)$$

Set

$$d_0 := \left(\frac{B_N}{2A_N |\text{Weyl}_g(\xi_0)|_g^2} \right)^{1/2} \quad (A_N \text{ and } B_N \text{ are positive constants defined in (4.3)}) \quad (3.2)$$

and let us choose

$$\tau_1, \dots, \tau_k \in \mathbb{R}^N \text{ with } \tau_i \neq \tau_j \text{ if } i \neq j \quad (3.3)$$

and for any $i = 1, \dots, k$

$$\mu_i = \varepsilon^\alpha (d_0 + d_i \varepsilon^\beta), \text{ where } d_1, \dots, d_k \in (0, +\infty), \quad \alpha := \frac{1}{2}, \quad \beta := \frac{N-6}{2N}. \quad (3.4)$$

Then, let us define

$$\begin{aligned} \mathcal{W}_i(z) := & \chi(d_g(z, \xi_0)) \mu_i^{-\frac{N-2}{2}} U \left(\frac{\exp_{\xi_0}^{-1}(z) - \varepsilon^\beta \tau_i}{\mu_i} \right) \\ & + \mu_i^2 \eta \left(\frac{|\exp_{\xi_0}^{-1}(z) - \varepsilon^\beta \tau_i|}{\mu_i} \right) \chi(d_g(z, \xi_0)) \mu_i^{-\frac{N-2}{2}} V \left(\frac{\exp_{\xi_0}^{-1}(z) - \varepsilon^\beta \tau_i}{\mu_i} \right), \quad z \in M \end{aligned} \quad (3.5)$$

where the functions U and V are defined, respectively, in (1.3) and (2.9). Set

$$\mathcal{C} := \{(\tau_1, \dots, \tau_k) \in \mathbb{R}^{kN} : \tau_i \neq \tau_j \text{ if } i \neq j\}.$$

We look for solutions of equation (1.4) or (2.3) of the form

$$u_\varepsilon(z) = \sum_{i=1}^k \mathcal{W}_i(z) + \phi_\varepsilon(z), \quad (3.6)$$

where the remainder term ϕ_ε belongs to the space \mathcal{K}^\perp defined as follows. For any $i = 1, \dots, k$ we introduce the functions

$$Z_{j,i}(z) = \chi(d_g(z, \xi_0)) \mu_i^{-\frac{N-2}{2}} \psi^j \left(\frac{\exp_{\xi_0}^{-1}(z) - \varepsilon^\beta \tau_i}{\mu_i} \right), \quad j = 0, 1, \dots, N, \quad (3.7)$$

where the functions ψ^j are defined in (2.8). We define the subspaces

$$\mathcal{K} := \text{Span} \{ \mathbf{i}^* (Z_{j,i}), j = 0, 1, \dots, N, i = 1, \dots, k \}$$

and

$$\mathcal{K}^\perp := \{ \phi \in H_g^1(M) : \langle \phi, \mathbf{i}^* (Z_{j,i}) \rangle = 0, j = 0, \dots, N, i = 1, \dots, k \}$$

and we also define the projections Π and Π^\perp of $H_g^1(M)$ onto \mathcal{K} and \mathcal{K}^\perp , respectively.

Therefore, equation 2.3 turns out to be equivalent to the system

$$\Pi^\perp \{ u_\varepsilon - \mathbf{i}^* [f(u_\varepsilon) - \varepsilon u_\varepsilon] \} = 0, \quad (3.8)$$

$$\Pi \{ u_\varepsilon - \mathbf{i}^* [f(u_\varepsilon) - \varepsilon u_\varepsilon] \} = 0. \quad (3.9)$$

where u_ε is given in (3.6).

3.2. The remainder term: solving the equation (3.8). In order to find the remainder term ϕ_ε we rewrite (3.8) as

$$\mathcal{E} + \mathcal{L}(\phi_\varepsilon) + \mathcal{N}(\phi_\varepsilon) = 0,$$

where the error term \mathcal{E} is defined by

$$\mathcal{E} := \Pi^\perp \left\{ \sum_{i=1}^k \mathcal{W}_i - \mathbf{i}^* \left[f \left(\sum_{i=1}^k \mathcal{W}_i \right) - \varepsilon \sum_{i=1}^k \mathcal{W}_i \right] \right\} \quad (3.10)$$

the linear operator \mathcal{L} is defined by

$$\mathcal{L}(\phi_\varepsilon) := \Pi^\perp \left\{ \phi_\varepsilon - \mathbf{i}^* \left[f' \left(\sum_{i=1}^k \mathcal{W}_i \right) \phi_\varepsilon - \varepsilon \phi_\varepsilon \right] \right\} \quad (3.11)$$

and the higher order term \mathcal{N} is defined by

$$\mathcal{N} := \Pi^\perp \left\{ -\mathbf{i}^* \left[f \left(\sum_{i=1}^k \mathcal{W}_i + \phi_\varepsilon \right) - f \left(\sum_{i=1}^k \mathcal{W}_i \right) - f' \left(\sum_{i=1}^k \mathcal{W}_i \right) \phi_\varepsilon \right] \right\}. \quad (3.12)$$

In order to solve equation (3.8), first of all we need to evaluate the $H_g^1(M)$ - norm of the error term \mathcal{E} . This is done in the following lemma whose proof is postponed in Section 4.

Lemma 3.1. *For any compact subset $A \subset (0, +\infty)^k \times \mathcal{C}$ there exists a positive constant C and $\varepsilon_0 > 0$ such that for any $(d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in A$ and for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\|\mathcal{E}\| \leq C \begin{cases} \varepsilon^{\frac{5}{4}} & \text{if } N = 7 \\ \varepsilon^{\frac{3}{2}} |\ln \varepsilon|^{\frac{5}{8}} & \text{if } N = 8 \\ \varepsilon^{\frac{3}{2}} & \text{if } N \geq 9. \end{cases} \quad (3.13)$$

Next, we need to understand the invertibility of the linear operators \mathcal{L} . This is done in the following lemma whose proof can be carried out as in [19].

Lemma 3.2. *For any compact subset $A \subset (0, +\infty)^k \times \mathcal{C}$ there exists a positive constant C and $\varepsilon_0 > 0$ such that for any $(d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in A$ and for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\|\mathcal{L}(\phi)\| \geq C \|\phi\| \text{ for any } \phi \in \mathcal{K}^\perp. \quad (3.14)$$

Finally, we are able to solve equation (3.8). This is done in the following proposition, whose proof is postponed in Section 4 and relies on a standard contraction mapping argument.

Proposition 3.1. *For any compact subset $A \subset (0, +\infty)^k \times \mathcal{C}$ there exists a positive constant C and ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ and for any $(d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in A$ there exists a unique function $\phi_\varepsilon \in \mathcal{K}^\perp$ which solves equation (3.8) such that*

$$\|\phi_\varepsilon\| \leq C \varepsilon^{\frac{3(N-2)}{2N} + \zeta} \quad (3.15)$$

for some $\zeta > 0$. Moreover, the map $(d_1, \dots, d_\ell, \tau_1, \dots, \tau_k) \rightarrow \phi_{\ell, \varepsilon}(d_1, \dots, d_\ell, \tau_1, \dots, \tau_k)$ is of class C^1 and

$$\|\nabla_{(d_1, \dots, d_\ell, \tau_1, \dots, \tau_k)} \phi_\varepsilon\| \leq C \varepsilon^{\frac{3(N-2)}{2N} + \zeta}$$

for some positive constants C and ζ .

3.3. The reduced problem: proof of Theorem 1.1. Let us introduce the reduced energy, defined by

$$\tilde{J}_\varepsilon(d_1, \dots, d_k, \tau_1, \dots, \tau_k) := J_\varepsilon \left(\sum_{i=1}^k \mathcal{W}_i + \phi_\varepsilon \right), \quad (d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in (0, +\infty)^k \times (\mathbb{R}^N)^k \quad (3.16)$$

where the remainder term ϕ_ε is defined in Proposition 3.1.

The following result allows as usual to reduce our problem to a finite dimensional one. The proof is standard and it is postponed in Section 4.

Proposition 3.2. (i) $\sum_{i=1}^k \mathcal{W}_i + \phi_\varepsilon$ is a solution to (1.4) if and only if $(d_1, \dots, d_k, \tau_1, \dots, \tau_k) \in (0, +\infty)^k \times (\mathbb{R}^N)^k$ is a critical point of the reduced energy (3.16)
(ii) The following expansion holds true

$$\tilde{J}_\varepsilon(d_1, \dots, d_k, \tau_1, \dots, \tau_k) := kD_N + c(\xi_0)\varepsilon^2 + \varepsilon^{3\frac{N-2}{N}} \mathfrak{J}(d_1, \dots, d_k, \tau_1, \dots, \tau_k) + o\left(\varepsilon^{3\frac{N-2}{N}}\right) \quad (3.17)$$

as $\varepsilon \rightarrow 0$, C^0 -uniformly with respect to $(d_1, \dots, d_k, \tau_1, \dots, \tau_k)$ in compact subsets of $(0, +\infty)^k \times \mathcal{C}$. Here $c(\xi_0) := k[-A_N |\text{Weyl}_g(\xi_0)|_g^2 d_0^4 + B_N d_0^2]$, A_N , B_N , D_N and E_N are positive constants defined in (4.3) and

$$\mathfrak{J}(d_1, \dots, d_k, \tau_1, \dots, \tau_k) := -\frac{1}{2}A_N d_0^4 \sum_{i=1}^k \mathcal{Q}(\xi_0)(\tau_i, \tau_i) - E_N d_0^{N-2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \frac{1}{|\tau_i - \tau_j|^{N-2}} - B_N \sum_{i=1}^k d_i^2. \quad (3.18)$$

Proof of Theorem 1.1. By (i) of Proposition (3.2), it is sufficient to find a critical point of the reduced energy \tilde{J}_ε . Now, the function \mathfrak{J} defined in (3.18), has a maximum point which is stable under C^0 -perturbations. Therefore, by (ii) of Proposition (3.2), we deduce that if ε is small enough there exists $(d_{1\varepsilon}, \dots, d_{k\varepsilon}, \tau_{1\varepsilon}, \dots, \tau_{k\varepsilon})$ critical point of \tilde{J}_ε . That concludes the proof. \square

4. APPENDIX

For any $i = 1, \dots, k$, we set

$$W_i(x) := \mu_i^{-\frac{N-2}{2}} U\left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i}\right) + \eta\left(\frac{|x - \varepsilon^\beta \tau_i|}{\mu_i}\right) \chi(d_g(z, \xi_0)) \mu_i^{-\frac{N-6}{2}} V\left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i}\right), \quad x \in \mathbb{R}^N.$$

It is important to point out that there exists $c > 0$ such that

$$|W_i(x)| \leq c \frac{\mu_i^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N-2}} \quad \forall x \in \mathbb{R}^N. \quad (4.1)$$

4.1. Proof of Lemma 3.1. It is easy to see that, $(\nu(\xi))$ is defined in (2.9)

$$\|\mathcal{E}\| \leq c \sum_{i=1}^k |-\Delta_g \mathcal{W}_i + (\beta_N R_g + \varepsilon) \mathcal{W}_i - \nu(\xi) Z_{0,i} - f(\mathcal{W}_i)|_{\frac{2N}{N+2}} + c \left| f\left(\sum_{i=1}^k \mathcal{W}_i\right) - \sum_{i=1}^k f(\mathcal{W}_i) \right|_{\frac{2N}{N+2}}$$

Arguing exactly as in Lemma 3.1 of [7], we can estimate each term

$$|-\Delta_g \mathcal{W}_i + (\beta_N R_g + \varepsilon) \mathcal{W}_i - \nu(\xi) Z_{0,i} - f(\mathcal{W}_i)|_{\frac{2N}{N+2}} = \begin{cases} O\left(\varepsilon^{\frac{5}{4}}\right) & \text{if } N = 7, \\ O\left(\varepsilon^{\frac{3}{2}} |\ln \varepsilon|^{\frac{5}{8}}\right) & \text{if } N = 8, \\ O\left(\varepsilon^{\frac{3}{2}}\right) & \text{if } N \geq 9. \end{cases}$$

Next, we show that

$$\left| f\left(\sum_{i=1}^k \mathcal{W}_i\right) - \sum_{i=1}^k f(\mathcal{W}_i) \right|_{\frac{2N}{N+2}} = O(\varepsilon^3).$$

Set for any $h = 1, \dots, k$ $B_h := B(\varepsilon^\beta \tau_h, \varepsilon^\beta \sigma/2)$ where $\sigma > 0$ and small enough. For (3.3) $B_h \subset B(0, r_0)$ and they are disjoint. We write

$$\begin{aligned} & \left| f\left(\sum_{i=1}^k \mathcal{W}_i\right) - \sum_{i=1}^k f(\mathcal{W}_i) \right|_{\frac{2N}{N+2}} \leq c \left[\int_{B(0, r_0)} (1 - \chi^{p+1}(|x|)) |\dots|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \\ & + c \left[\int_{B(0, r_0) \setminus \cup_h B_h} |\dots|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} + c \sum_{h=1}^k \left[\int_{B_h} |\dots|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \\ & \leq c \sum_{i=1}^k \left[\int_{B(0, r_0)} (1 - \chi^{p+1}(|x|)) |W_i|^{\frac{2N}{N-2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} + c \left[\int_{B(0, r_0) \setminus \cup_h B_h} |W_i|^{\frac{2N}{N-2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \\ & + c \sum_{h=1}^k \left[\int_{B_h} \left| W_h^{p-1} \sum_{i \neq h} W_i \right|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} + c \sum_{h=1}^k \left[\int_{B_h} \left| \sum_{i \neq h} W_i \right|^{\frac{2N}{N-2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}}. \end{aligned}$$

Let us estimate each term in the previous expression. We use (4.1).

$$\begin{aligned} & \sum_{i=1}^k \left[\int_{B(0, r_0)} (1 - \chi^{p+1}(|x|)) |W_i|^{\frac{2N}{N-2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \leq c \sum_{i=1}^k \left[\int_{\mathbb{R}^N \setminus B(0, r_0)} \frac{\mu_i^N}{|x - \varepsilon^\beta \tau_i|^{2N}} dx \right]^{\frac{N+2}{2N}} \\ & \leq c \sum_{i=1}^k \frac{\mu_i^{\frac{N+2}{2}}}{\varepsilon^{\beta \frac{N+2}{2}}} \left[\int_{\mathbb{R}^N \setminus B(0, r_0/\varepsilon^\beta)} \frac{1}{|y - \tau_i|^{2N}} dy \right]^{\frac{N+2}{2N}} \\ & \leq c \varepsilon^{(\alpha-\beta) \frac{N+2}{2} + \alpha \frac{N+2}{2}} \leq c \varepsilon^{3 \frac{N+2}{2N}}, \\ & \left[\int_{B(0, r_0) \setminus \cup_h B_h} |W_i|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \leq c \frac{\mu_i^{\frac{N+2}{2}}}{\varepsilon^{\beta \frac{N+2}{2}}} \left[\int_{B(0, r_0/\varepsilon^\beta) \setminus \cup_h B(\tau_h, \sigma/2)} \frac{1}{|y - \tau_i|^{2N}} dy \right]^{\frac{N+2}{2N}} \\ & \leq C \varepsilon^{(\alpha-\beta) \frac{N+2}{2}} \leq C \varepsilon^{3 \frac{N+2}{2N}}, \\ & \sum_{h=1}^k \left[\int_{B_h} \left| W_h^{p-1} \sum_{i \neq h} W_i \right|^{\frac{2N}{N+2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \leq c \sum_{h=1}^k \sum_{i \neq h} \left[\int_{B_h} \frac{\mu_h^{\frac{4N}{N+2}}}{|x - \varepsilon^\beta \tau_h|^{\frac{8N}{N+2}}} \frac{\mu_i^{\frac{N(N-2)}{N+2}}}{|x - \varepsilon^\beta \tau_i|^{\frac{2N(N-2)}{N+2}}} dx \right]^{\frac{N+2}{2N}} \\ & \leq c \sum_{h=1}^k \sum_{i \neq h} \mu_h^2 \mu_i^{\frac{N-2}{2}} \left[\int_{B_h} \frac{1}{|x - \varepsilon^\beta \tau_h|^{\frac{8N}{N+2}}} dx \right]^{\frac{N+2}{2N}} \\ & \leq c \sum_{h=1}^k \sum_{i \neq h} \mu_h^2 \mu_i^{\frac{N-2}{2}} \left[\int_{B(0, \varepsilon^\beta \sigma/2)} \frac{1}{|y|^{\frac{8N}{N+2}}} dy \right]^{\frac{N+2}{2N}} \\ & \leq c \sum_{h=1}^k \sum_{i \neq h} \mu_h^2 \mu_i^{\frac{N-2}{2}} \varepsilon^{\beta \frac{N-6}{2}} \leq c \varepsilon^{3 \frac{N+2}{2N}} \end{aligned}$$

and

$$\begin{aligned} & \sum_{h=1}^k \left[\int_{B_h} \left| \sum_{i \neq h} W_i \right|^{\frac{2N}{N-2}} |g(x)|^{\frac{1}{2}} dx \right]^{\frac{N+2}{2N}} \leq c \sum_{h=1}^k \sum_{i \neq h} \left[\int_{B_h} \frac{\mu_i^N}{|x - \varepsilon^\beta \tau_i|^{2N}} dx \right]^{\frac{N+2}{2N}} \\ & \leq C \frac{\mu_i^{\frac{N+2}{2}}}{\varepsilon^{\beta \frac{N+2}{2}}} \left[\int_{B(\tau_h, \sigma/2)} \frac{1}{|y - \tau_i|^{2N}} dy \right]^{\frac{N+2}{2N}} \leq c \varepsilon^{3 \frac{N+2}{2N}}. \end{aligned}$$

4.2. Proof of Proposition 3.2. It is quite standard to prove that

$$J_\varepsilon \left(\sum_{i=1}^k \mathcal{W}_i + \phi_\varepsilon \right) = J_\varepsilon \left(\sum_{i=1}^k \mathcal{W}_i \right) + \Theta$$

C^0 – uniformly with respect to $(d_1, \dots, d_k, \tau_1, \dots, \tau_k)$ in compact subset of $(0, +\infty)^k \times \mathcal{C}$, where Θ is a smooth function such that $|\Theta|, |\nabla \Theta| = O(\varepsilon^{3\frac{N-2}{N}+\zeta})$ for some small $\zeta > 0$. We shall prove that

$$\begin{aligned} J_\varepsilon \left(\sum_{i=1}^k \mathcal{W}_i \right) &= kD_N + k\varepsilon^2 \left[-A_N |\text{Weyl}_g(\xi_0)|_g^2 + B_N d_0^2 \right] \\ &+ \varepsilon^{3\frac{N-2}{N}} \left[-\frac{1}{2} A_N d_0^4 \sum_{i=1}^k \mathcal{Q}(\xi_0)(\tau_i, \tau_i) - E_N d_0^{N-2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \frac{1}{|\tau_i - \tau_j|^{N-2}} - B_N \sum_{i=1}^k d_i^2 \right] + \Theta, \end{aligned} \quad (4.2)$$

where

$$A_N := \frac{K_N^{-N}}{24N(N-4)(N-6)}, \quad B_N := \frac{2(N-1)K_N^{-N}}{N(N-2)(N-4)}, \quad D_N := \frac{K_N^{-N}}{N}, \quad E_N := \alpha_N \int_{\mathbb{R}^N} U^p(y) dy \quad (4.3)$$

and K_N is the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. Here Θ is a smooth function such that $|\Theta|, |\nabla \Theta| = O(\varepsilon^{3\frac{N-2}{N}+\zeta})$ for some small $\zeta > 0$.

Let us prove (4.2).

$$\begin{aligned} J_\varepsilon \left(\sum_{i=1}^k \mathcal{W}_i \right) &= \underbrace{\sum_{i=1}^k J_\varepsilon(\mathcal{W}_i)}_I - \underbrace{\sum_{j < i} \int_M f(\mathcal{W}_i) \mathcal{W}_j d\nu_g}_{II} \\ &+ \sum_{i < j} \int_M [\nabla_g \mathcal{W}_i \nabla_g \mathcal{W}_j + \beta_N R_g \mathcal{W}_i \mathcal{W}_j - f(\mathcal{W}_i) \mathcal{W}_j] d\nu_g \\ &- \int_M \left[F \left(\sum_{i=1}^k \mathcal{W}_i \right) - \sum_{i=1}^k F(\mathcal{W}_i) - \sum_{i \neq j} f(\mathcal{W}_i) \mathcal{W}_j \right] d\nu_g + \varepsilon \sum_{i < j} \int_M \mathcal{W}_i \mathcal{W}_j d\nu_g. \end{aligned} \quad (4.4)$$

First of all, we estimate the two leading terms I and II in (4.4).

The term I is given by the contribution of each bubble. Indeed, in Section 4 of [7] it was proved that for any $i = 1, \dots, k$

$$J_\varepsilon(\mathcal{W}_i) = D_N - A_N |\text{Weyl}_g(\xi_i)|_g^2 \mu_i^4 + \varepsilon B_N \mu_i^2 + \begin{cases} O(\varepsilon^{\frac{5}{2}}) & \text{if } N = 7, \\ O(\varepsilon^3 |\ln \varepsilon|^3) & \text{if } N = 8, \\ O(\varepsilon^3) & \text{if } N \geq 9 \end{cases}. \quad (4.5)$$

Now, by the choice of d_0 in (3.2) and the choice of μ_i , α and β in (3.4), we get

$$\begin{aligned} |\text{Weyl}_g(\xi_i)|^2 &= |\text{Weyl}_g(\xi_0)|^2 + \frac{1}{2} \mathcal{Q}(\xi_0) [\tau_i, \tau_i] \varepsilon^{2\beta} + O(\varepsilon^{3\beta}), \\ \mu_i^4 &= \varepsilon^{4\alpha} \left[d_0^4 + 4d_0^3 d_i \varepsilon^\beta + 6d_0^2 d_i^2 \varepsilon^{2\beta} + O(\varepsilon^{3\beta}) \right], \\ \mu_i^2 &= \varepsilon^{2\alpha} \left[d_0^2 + 2d_0 d_i \varepsilon^\beta + d_i^2 \varepsilon^{2\beta} \right]. \end{aligned}$$

Therefore, a straightforward computation shows that

$$\begin{aligned} -A_N |\text{Weyl}_g(\xi_i)|_g^2 \mu_i^4 + \varepsilon B_N \mu_i^2 &= \varepsilon^2 \left[-A_N |\text{Weyl}_g(\xi_0)|_g^2 + B_N d_0^2 \right] \\ &+ \varepsilon^{3\frac{N-2}{N}} \left[-\frac{1}{2} A_N d_0^4 \mathcal{Q}(\xi_0)(\tau_i, \tau_i) - B_N \sum_{i=1}^k d_i^2 \right] + O\left(\varepsilon^{\frac{7N-18}{2N}}\right). \end{aligned} \quad (4.6)$$

By (4.5) and (4.6) we deduce the estimate of I .

The term II is given by the interaction of different bubbles. For any $h = 1, \dots, k$ let $B_h := B(\varepsilon^\beta \tau_h, \varepsilon^\beta \sigma/2)$. By (3.3) we deduce that $B_h \subset B(0, r_0)$ provided σ is small enough and they are disjoint. Therefore, if

$i \neq j$

$$\begin{aligned}
\int_M f(W_i) \mathcal{W}_j d\nu_g &= \int_{B_i} f(W_i(x)) W_j(x) |g(x)|^{1/2} dx + \int_{B(0, r_0) \setminus B_i} f(W_i(x)) W_j(x) |g(x)|^{1/2} dx \\
&\quad + \int_{B(0, r_0)} [1 - \chi^{p+1}(|x|)] f(W_i(x)) W_j(x) |g(x)|^{1/2} dx \\
&= E_N d_0^{N-2} \frac{1}{|\tau_i - \tau_j|^{N-2}} \varepsilon^{3 \frac{N-2}{N}} + O(\varepsilon^3). \tag{4.7}
\end{aligned}$$

Indeed, the main term of (4.7) is given by

$$\begin{aligned}
&\int_{B_i} f(W_i(x)) W_j(x) |g(x)|^{1/2} dx \\
&= \int_{B_i} f \left(\mu_i^{-\frac{N-2}{2}} U \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) + \mu_i^{-\frac{N-6}{2}} \eta \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) V \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) \right) \times \\
&\quad \times \left(\mu_j^{-\frac{N-2}{2}} U \left(\frac{x - \varepsilon^\beta \tau_j}{\mu_j} \right) + \mu_j^{-\frac{N-6}{2}} \eta \left(\frac{x - \varepsilon^\beta \tau_j}{\mu_j} \right) V \left(\frac{x - \varepsilon^\beta \tau_j}{\mu_j} \right) \right) |g(x)|^{1/2} dx \\
&\quad (\eta \left(\frac{x - \varepsilon^\beta \tau_j}{\mu_j} \right) = 0 \text{ if } x \in B_i \text{ and } \varepsilon \text{ is small enough}) \\
&= \int_{B_i} f \left(\mu_i^{-\frac{N-2}{2}} U \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) + \mu_i^{-\frac{N-6}{2}} \eta \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) V \left(\frac{x - \varepsilon^\beta \tau_i}{\mu_i} \right) \right) \times \\
&\quad \times \left(\mu_j^{-\frac{N-2}{2}} U \left(\frac{x - \varepsilon^\beta \tau_j}{\mu_j} \right) \right) |g(x)|^{1/2} dx \\
&\quad (\text{setting } x - \varepsilon^\beta \tau_i = \mu_i y) \\
&= \mu_i^{\frac{N-2}{2}} \int_{B(0, \varepsilon^\beta \sigma / 2 \mu_i)} f(U(y) + \mu_i^2 \eta(|y|) V(y)) \frac{\alpha_N \mu_j^{\frac{N-2}{2}}}{(\mu_j^2 + |\mu_j y + \varepsilon^\beta (\tau_i - \tau_j)|^2)^{\frac{N-2}{2}}} \times \\
&\quad \times |g(\mu_i y + \varepsilon^\beta \tau_i)|^{1/2} dy \\
&= \alpha_N \frac{\mu_i^{\frac{N-2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{\beta(N-2)} |\tau_i - \tau_j|^{N-2}} \left[\int_{\mathbb{R}^N} U^p(y) dy + O(\mu_i^2) + O\left(\left(\frac{\mu_i}{\varepsilon^\beta}\right)^N\right) \right] \\
&= \varepsilon^{3 \frac{N-2}{N}} \frac{d_0^{N-2}}{|\tau_i - \tau_j|^{N-2}} \alpha_N \int_{\mathbb{R}^N} U^p(y) dy + O(\varepsilon^3), \tag{4.8}
\end{aligned}$$

because of the choice of μ_i in (3.4). Moreover, by (4.1)

$$\begin{aligned}
&\left| \int_{B(0, r_0) \setminus B_i} f(W_i(x)) W_j(x) |g(x)|^{1/2} dx \right| \\
&\leq c \int_{\mathbb{R}^N \setminus B_i} \frac{\mu_i^{\frac{N+2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N+2}} \frac{\mu_j^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_j|^{N-2}} dx \quad (\text{setting } x = \varepsilon^\beta y) \\
&\leq c \frac{\mu_i^{\frac{N+2}{2}} \mu_j^{\frac{N-2}{2}}}{\beta^N} \int_{\mathbb{R}^N \setminus B(\tau_i, \sigma/2)} \frac{1}{|y - \tau_i|^{N+2}} \frac{\mu_j^{\frac{N-2}{2}}}{|y - \tau_j|^{N-2}} dy \\
&= O\left(\frac{\mu_i^{\frac{N+2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{\beta N}}\right) = O(\varepsilon^3)
\end{aligned}$$

and

$$\left| \int_{B(0,r_0)} [1 - \chi^{p+1}(|x|)] f(W_i(x)) W_j(x) |g(x)|^{1/2} dx \right| = O\left(\varepsilon^{\frac{N}{2}}\right).$$

Finally, let us prove that all the other terms in (4.4) are of higher order.

By (2.5) and (2.6), we deduce that

$$|\Delta_g \mathcal{W}_i + \beta_N \mathbf{R}_g \mathcal{W}_i - f(\mathcal{W}_i)|(\exp_{\xi_0}(x)) \leq c \frac{\mu_i^{\frac{N-2}{2}}}{(\mu_i^2 + |x - \varepsilon^\beta \tau_i|^2)^{\frac{N-2}{2}}}$$

and so by (4.1) if $i \neq j$ we have

$$\begin{aligned} & \left| \int_M [\nabla_g \mathcal{W}_i \nabla_g \mathcal{W}_j + \beta_N \mathbf{R}_g \mathcal{W}_i \mathcal{W}_j - f(\mathcal{W}_i) \mathcal{W}_j] d\nu_g \right| \\ &= \left| \int_M [\Delta_g \mathcal{W}_i + \beta_N \mathbf{R}_g \mathcal{W}_i - f(\mathcal{W}_i)] \mathcal{W}_j d\nu_g \right| \\ &\leq c \int_{B(0,r_0)} \frac{\mu_i^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N-2}} \frac{\mu_j^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_j|^{N-2}} dx \text{ (setting } x = \varepsilon^\beta y) \\ &\leq c \frac{\mu_i^{\frac{N-2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{\beta N-4}} \int_{\mathbb{R}^N} \frac{1}{|y - \tau_i|^{N-2}} \frac{1}{|y - \tau_j|^{N-2}} dx = O(\varepsilon^3). \end{aligned}$$

Moreover, if $i \neq j$

$$\begin{aligned} & \left| \int_M \mathcal{W}_i \mathcal{W}_j d\nu_g \right| = \left| \int_{B(0,r_0)} W_i(x) W_j(x) |g(x)|^{1/2} dx \right| \\ &\leq c \int_{B(0,r_0)} \frac{\mu_i^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N-2}} \frac{\mu_j^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_j|^{N-2}} dx \text{ (setting } x = \varepsilon^\beta y) \\ &\leq c \frac{\mu_i^{\frac{N-2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{\beta N-4}} \int_{\mathbb{R}^N} \frac{1}{|y - \tau_i|^{N-2}} \frac{1}{|y - \tau_j|^{N-2}} dx = O(\varepsilon^3). \end{aligned}$$

Finally, we have

$$\begin{aligned} & \int_M \left[F\left(\sum_{i=1}^k \mathcal{W}_i\right) - \sum_{i=1}^k F(\mathcal{W}_i) - \sum_{i \neq j} f(\mathcal{W}_i) \mathcal{W}_j \right] d\nu_g \\ &= \sum_{h=1}^k \int_{B_h} \left[F\left(\sum_{i=1}^k W_i\right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right] |g(x)|^{1/2} dx \\ &+ \int_{B(0,r_0) \setminus \cup_h B_h} \left[F\left(\sum_{i=1}^k W_i\right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right] |g(x)|^{1/2} dx \\ &+ \int_{B(0,r_0)} [1 - \chi^{p+1}(|x|)] \left[F\left(\sum_{i=1}^k W_i\right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right] |g(x)|^{1/2} dx. \end{aligned}$$

It is immediate that

$$\int_{B(0,r_0)} [1 - \chi^{p+1}(|x|)] \left[F \left(\sum_{i=1}^k W_i \right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right] |g(x)|^{1/2} dx = O \left(\varepsilon^{\frac{N}{2}} \right).$$

Moreover, outside the k balls we get

$$\begin{aligned} & \int_{B(0,r_0) \setminus \cup_h B_h} \left| F \left(\sum_{i=1}^k W_i \right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right| |g(x)|^{1/2} dx \\ & \leq c \sum_{i \neq j} \int_{B(0,r_0) \setminus \cup_h B_h} (|W_i|^{p-1} W_j^2 + |W_j|^{p-1} W_i^2) dx = O(\varepsilon^3), \end{aligned}$$

because if $2 < q = p + 1 < 3$

$$|(a+b)^q - a^q - b^q - qa^{q-1}b - qab^{q-1}| \leq c(a^2b^{q-2} + a^{q-2}b^2) \text{ for any } a, b > 0$$

and if $j \neq i$

$$\begin{aligned} & \int_{B(0,r_0) \setminus \cup_h B_h} |W_i|^{p-1} W_j^2 dx \leq c \int_{B(0,r_0) \setminus \cup_h B_h} \frac{\mu_i^2}{|x - \varepsilon^\beta \tau_i|^4} \frac{\mu_j^{N-2}}{|x - \varepsilon^\beta \tau_j|^{2(N-2)}} dx \text{ (setting } x = \varepsilon^\beta y) \\ & \leq c \frac{\mu_i^2 \mu_j^{N-2}}{\varepsilon^{\beta N}} \int_{\mathbb{R}^N \setminus \cup_h B(\tau_h, \sigma/2)} \frac{1}{|y - \tau_i|^4} \frac{1}{|y - \tau_j|^{2(N-2)}} dy. \end{aligned}$$

On each ball B_h we also have

$$\begin{aligned} & \int_{B_h} \left| F \left(\sum_{i=1}^k W_i \right) - \sum_{i=1}^k F(W_i) - \sum_{i \neq j} f(W_i) W_j \right| |g(x)|^{1/2} dx \\ & \leq \int_{B_h} \left| F \left(W_h + \sum_{i \neq h} W_i \right) - F(W_h) - \sum_{j \neq h} f(W_h) W_j \right| dx \\ & \quad + \sum_{i \neq h} \int_{B_h} |F(W_i)| dx + \sum_{\substack{i \neq h \\ j \neq i}} \int_{B_h} |f(W_i) W_j| dx \\ & \leq c \sum_{i \neq h} \int_{B_h} W_h^{p-1} W_i^2 dx + c \sum_{i \neq h} \int_{B_h} W_i^{p+1} dx + c \sum_{\substack{i \neq h \\ j \neq i}} \int_{B_h} W_i^p W_j dx, \end{aligned} \tag{4.9}$$

because if $q = p + 1 \geq 1$

$$|(a+b)^q - a^q - b^q - qa^{q-1}b| \leq c(b^q + a^{q-2}b^2) \text{ for any } a, b > 0.$$

Now we use (4.1) and we get if $i \neq h$

$$\begin{aligned} & \int_{B_h} W_h^{p-1} W_i^2 dx \leq c \int_{B_h} \frac{\mu_h^2}{|x - \varepsilon^\beta \tau_h|^4} \frac{\mu_i^{N-2}}{|x - \varepsilon^\beta \tau_i|^{2(N-2)}} dx \text{ (setting } x = \varepsilon^\beta y) \\ & \leq c \frac{\mu_h^2 \mu_i^{N-2}}{\varepsilon^{\beta N}} \int_{B(\tau_h, \sigma/2)} \frac{1}{|y - \tau_h|^4} \frac{1}{|y - \tau_i|^{2(N-2)}} dy = O(\varepsilon^3), \end{aligned}$$

if $j, i \neq h$

$$\begin{aligned} & \int_{B_h} |W_i|^p W_j dx \leq c \int_{B_h} \frac{\mu_i^{\frac{N+2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N+2}} \frac{\mu_j^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_j|^{N-2}} dx \\ & \leq c \frac{\mu_i^{\frac{N+2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{2\beta N}} |B_h| \leq c \frac{\mu_i^{\frac{N+2}{2}} \mu_j^{\frac{N-2}{2}}}{\varepsilon^{\beta N}} = O(\varepsilon^3), \end{aligned}$$

if $i \neq h$

$$\begin{aligned} \int_{B_h} |W_i|^p W_h dx &\leq c \int_{B_h} \frac{\mu_i^{\frac{N+2}{2}}}{|x - \varepsilon^\beta \tau_i|^{N+2}} \frac{\mu_h^{\frac{N-2}{2}}}{|x - \varepsilon^\beta \tau_h|^{N-2}} dx \\ &\leq c \frac{\mu_i^{\frac{N-2}{2}}}{\varepsilon^{\beta(N+2)}} \int_{B_h} \frac{1}{|x - \varepsilon^\beta \tau_h|^{N-2}} dx \leq c \frac{\mu_i^{\frac{N-2}{2}} \mu_h^{\frac{N-2}{2}}}{\varepsilon^{\beta N}} = O(\varepsilon^3) \end{aligned}$$

and if $i \neq h$

$$\int_{B_h} W_i^{p+1} dx \leq c \int_{B_h} \frac{\mu_i^N}{|x - \varepsilon^\beta \tau_i|^{2N}} dx \leq c \frac{\mu_i^N}{\varepsilon^{2\beta N}} |B_h| \leq c \frac{\mu_i^N}{\varepsilon^{\beta N}} = O(\varepsilon^3).$$

That concludes the proof.

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ANGELA PISTOIA, DIPARTIMENTO DI SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA, SAPIENZA UNIVERSITÀ DI ROMA,
VIA ANTONIO SCARPA 16, 00161 ROMA, ITALY

E-mail address: `angela.pistoia@uniroma1.it`

GIUSI VAIRA, DIPARTIMENTO DI SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA, SAPIENZA UNIVERSITÀ DI ROMA,
VIA ANTONIO SCARPA 16, 00161 ROMA, ITALY

E-mail address: `vaira.giusi@gmail.com`